

Deformed General Relativity

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Abstract

Deformed special relativity is embedded in deformed general relativity using the methods of canonical relativity and loop quantum gravity. Phase-space dependent deformations of symmetry algebras then appear, which in some regimes can be rewritten as non-linear Poincaré algebras with momentum-dependent deformations of commutators between boosts and time translations. In contrast to deformed special relativity, the deformations are derived for generators with an unambiguous physical role, following from the relationship between canonical constraints of gravity with stress-energy components. The original deformation does not appear in momentum space and does not give rise to non-locality issues or problems with macroscopic objects. Contact with deformed special relativity may help to test loop quantum gravity or restrict its quantization ambiguities.

1 Introduction

A quantum theory of gravity combines the fundamental constants of nature G and \hbar , characteristic of the ingredients of general relativity and quantum mechanics. It should therefore assign a specific role to the Planck length $\ell_P = \sqrt{G\hbar}$, the Planck mass $m_P = \hbar/\ell_P$ or the Planck density $\rho_P = m_P/\ell_P^3$ beyond what one may expect on purely dimensional grounds. One suggestion that is often made is the presence of an invariant length, ℓ_P , on the same footing as the invariant speed of light c in special and general relativity. Even more specifically, ℓ_P may pose a lower limit to distances, or ρ_P an upper limit to densities, or m_P an upper limit to masses and energies. With these assumptions, especially the last one, one can be led to different versions of deformed special relativity [1, 2, 3, 4], based on deformations of the Poincaré algebra so that a second invariant constant is introduced.

While an invariant shortest distance or largest density may sound natural in quantum gravity, it is by no means implied just by the fact that G and \hbar both appear in the theory. Setting aside the question of bounds, it is not even clear whether there should be an invariant distance or mass. While the ingredients G and \hbar of ℓ_P and m_P , and so the Planck quantities themselves, must be invariant under transformations of reference frames, the question of invariant distances or masses depends on what role ℓ_P and m_P

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play for physical observables. The question of observables or measurement procedures is complicated in any combination of quantum physics with general relativity, and therefore detailed knowledge (rather than just dimensional expectations) is required before these questions can be answered.

What gives substance to the claims of deformed special relativity is the fact that symmetries and their quantum realizations present some of the most fundamental concepts in physics. The mathematical rigidity of possible deformations of the Poincaré algebra by quantum corrections or other effects allows interesting tests of the current understanding of quantum gravity in general terms, or, if such effects are derived from one of the candidate theories, means to compare the different, usually disparate approaches. In this article, we take this viewpoint and have a general look at canonical quantum gravity.¹

2 From Poincaré transformations to hypersurface deformations

When combined with gravity, space-time described by special relativity is too limited. One should rather use general relativity and its richer structure of arbitrary coordinate transformations. Deformed special relativity, in which general covariance is not realized but a non-zero gravitational constant (and in some arguments gravitational phenomena such as black holes) is assumed can be considered only as a limit. But it is not clear whether there is a consistent relativistic procedure that does away with general covariance but still keeps the gravitational constant as a fundamental parameter. For this reason, we propose to ask the question of possible deformations for the symmetries underlying general relativity. In algebraic form, we go from the well-known Poincaré relations²

$$\{P_\mu, P_\nu\} = 0 \tag{1}$$

$$\{M_{\mu\nu}, P_\rho\} = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu \tag{2}$$

$$\{M_{\mu\nu}, M_{\rho\sigma}\} = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} \tag{3}$$

¹Our discussions and results are different from earlier attempts to derive deformed special relativity or Lorentz violations from quantum gravity. In [5], 2 + 1-dimensional models have given rise to deformed Poincaré algebras, but the key property (non-vanishing Poisson brackets of gravitational connection components) is not necessarily realized in 3 + 1 dimensions. Another suggestion made in [5] for 3 + 1 dimensions makes use of Chern–Simons-like boundary terms of Plebanski actions, whose algebra turns out to be deformed [6]. We will make use of boundary observables as well, but already the bulk terms of canonical gauge generators will obey a deformed algebra. More recently, in [7] methods related to quantum field theory on a modified space-time background have been used, but no clear deformation or violation effects have been found. Moreover, the latter analysis ignored a consistent treatment of quantum space-time structures, the key ingredient used here.

²We will always use commutators of classical type, computed as Poisson brackets $\{\cdot, \cdot\}$ of phase-space functions. A quantum analog of operators would then read $[\cdot, \cdot] = i\hbar\{\cdot, \cdot\}$. Although we will introduce quantum effects as crucial ingredients for deformations, they will be computed from effective equations. We therefore write $\{\cdot, \cdot\}$ instead of $[\cdot, \cdot]$ to avoid the impression that we are requantizing terms of effective equations.

to the much more unwieldy algebra of hypersurface deformations in space-time.

2.1 Hypersurface-deformation algebra

According to Dirac [8], a canonical field theory on space-time foliated by equal-time slices is generally covariant if it is invariant under the hypersurface-deformation algebra

$$\{D[M^a], D[N^a]\} = D[\mathcal{L}_{N^b} M^a] \quad (4)$$

$$\{H[M], D[N^a]\} = H[\mathcal{L}_{N^b} M] \quad (5)$$

$$\{H[M], H[N]\} = D[h^{ab}(M\nabla_b N - N\nabla_b M)] \quad (6)$$

whose generators $D[N^a]$ and $H[N]$ depend on shift vector fields N^a and lapse functions N on the spatial slices. (For an introduction to methods and properties of canonical gravity used in this paper we refer to [9].) Also the metric h_{ab} on spatial slices, or its inverse h^{ab} , appears in the structure functions. If $D[N^a]$ and $H[N]$ are realized as gauge generators, an infinitesimal space-time diffeomorphism along a vector field ξ^μ is represented by the gauge transformation $\delta_{\epsilon^\mu} f = \{f, H[\epsilon] + D[\epsilon^a]\}$ on phase-space functions, with $\epsilon = N\xi^0$ and $\epsilon^a = \xi^a + N^a \xi^0$ [10]. The hypersurface-deformation algebra therefore describes general covariance, just as the Poincaré algebra describes the symmetries of special relativity.

The generators $D[N^a]$ and $H[N]$ consist of bulk terms which vanish as the canonical constraints, and spatial boundary terms if they are computed for finite regions or in space-times with specific asymptotic fall-off conditions. Boundary terms are not required to vanish and provide energy and (angular) momentum observables, for finite regions of Brown–York type [11] and for asymptotic regions of ADM type [12]. The generators are therefore physically related to those of the Poincaré algebra, just as the transformations are geometrically related. In both views, however, the freedom in the algebra is much larger for hypersurface deformations, which are not required to be linear, and for their generators, whose physical expressions as energy and momentum refer to a large set of observers in different states of motion depending on which bounded or asymptotic region is chosen.

The hypersurface-deformation algebra differs from the Poincaré algebra in several important respects, not only in the fact that it is much larger and in fact infinite-dimensional. While both algebras depend on the metric, these coefficients in the case of the Poincaré algebra (2) and (3) are constants because they just refer to Minkowski space-time. The spatial metric in (6), on the other hand, in general depends on the position and is a spatial tensor. The Minkowski metric in the Poincaré algebra determines structure constants; the spatial metric on a slice in a curved space-time used in the hypersurface-deformation algebra determines structure functions. The hypersurface-deformation algebra is not a Lie algebra, but a Lie algebroid [13]. Its deformations, in contrast to the Poincaré algebra, have not been studied systematically, and therefore it presents an interesting, more-general object in the context of deformed relativity. By its relation to general covariance, it automatically incorporates gravity.

If there are reasons to believe that the Poincaré algebra is deformed, there should be a corresponding deformed version of the hypersurface-deformation algebra, to make sure that the gravitational force can be described consistently under the deformation. Vice versa, if there is a deformation of the hypersurface-deformation algebra,³ it entails a deformation of the Poincaré algebra. While it is difficult to embed deformed Poincaré algebras in a hypersurface-deformation algebra, deriving a deformed Poincaré algebra from a deformed hypersurface-deformation algebra can be accomplished by restricting the algebra to linear functions N^a and N in a given set of coordinates, together with Euclidean spatial slices such that $h_{ab} = \delta_{ab}$. Choosing

$$N(x) = \Delta t + v_a x^a \quad , \quad N^a(x) = \Delta x^a + R_b^a x^b$$

relates hypersurface deformations to Poincaré transformations with time translation Δt , spatial translations Δx^a , boosts by v_a and spatial rotations with matrices R_b^a . We call the resulting Poincaré-type algebra the *linear limit* of the (deformed) hypersurface-deformation algebra we start with.

In the presence of deformed algebras, the corresponding space-time structure differs from the classical one: gauge transformations do not agree with Lie derivatives, and any dynamics consistent with a deformed algebra differs from general relativity [14, 15, 16].⁴ The identification of a deformed Poincaré algebra as a restriction of the hypersurface-deformation algebra with linear N^a and N may therefore seem ambiguous. However, even without reference to classical space-times, the linear limit is distinguished. Linear N and N^a in (4)–(7) lead to the only closed subalgebra if h^{ab} is constant. Therefore, if there is a deformed Poincaré algebra, it can only be the linear limit of the hypersurface-deformation algebra.

For the linear limit to be meaningful, we assume, as always in special-relativistic situations, that all energies and momenta involved are sufficiently small and that their back-reaction on space-time can be ignored. Using a constant $h_{ab} = \delta_{ab}$ is then justified. If back-reaction cannot be ignored, there is simply no special-relativity limit of the theory.

2.2 Deformations

Several examples of deformed hypersurface-deformation algebras have been found in loop quantum gravity, using effective methods and operator calculations. All these deformations leave the D -relations (4) and (5) unchanged, while (6) is modified to

$$\{H[M], H[N]\} = D[\beta h^{ab}(M\nabla_b N - N\nabla_b M)] \quad (7)$$

³In phrases like this, we use “deformation” in two different meanings. The context makes it clear which is implied.

⁴There is no “effective line element” in such a situation: Gauge transformations of h_{ab} do not match with coordinate transformations of dx^a to give an invariant ds^2 . Nevertheless, all observables of interest can be computed by canonical methods. Comparisons with deformed Poincaré algebras in the linear limit may suggest corresponding quantum space-time models, such as κ -Minkowski. We will come back to this question at the end of this article.

with a phase-function $\beta \neq 1$ that may depend on the spatial metric or extrinsic curvature. Loop quantum gravity therefore confirms the expectation that quantum geometry should lead to deformations of symmetry algebras of space-time. In fact, no undeformed consistent version of symmetries at this quantum level has been found.

There is a broad consensus in loop quantum gravity that *off-shell* constrained algebras must be deformed if quantum-geometry effects of the theory are included. (See [16] for a detailed list of models.) The first such deformations have been found by effective methods in models of perturbative inhomogeneity [17] and in spherical symmetry [18], in both cases using inverse-triad corrections [19, 20]. A second type of corrections, holonomy corrections, has been implemented consistently in the same type of models [21, 22]. Analogous deformations appear in operator calculations of the constraint algebra for $2+1$ -dimensional models, with holonomy corrections [23] and inverse-triad corrections [24, 25, 26]. (An especially striking feature of holonomy corrections is that they trigger signature change at high density [16, 27].)

In all cases, the algebra is deformed in the same way, with characteristic functions β depending on extrinsic curvature for holonomy corrections and on the spatial metric for inverse-triad corrections. Unmodified space-time structures appear only in cases in which the classical structure is presupposed by fixing the gauge before quantization, a procedure which in cosmology (and elsewhere) is known to lead to incorrect results. Deformed hypersurface-deformation algebras are therefore an unavoidable consequence of the quantization steps undertaken in loop quantum gravity, in particular the use of holonomies as basic operators [28, 29]. Loop quantum gravity leads to deformed space-time structures and to deformed general relativity in a semiclassical limit.

All quantum corrections are state-dependent and must therefore be parameterized suitably, given that knowledge of quantum-gravity states is limited. Inverse-triad corrections in loop quantum gravity lead to a deformation function β depending on the size of discrete plaquettes relative to the Planck area, and therefore on the spatial metric. Holonomy corrections depend on the momentum of the spatial metric, related to extrinsic curvature or the time derivative of the spatial metric.⁵ Without using detailed expressions which can be derived in loop quantum gravity, one can easily expect corrections of these two types. Inverse-triad corrections incorporate implications of discrete space, while holonomy corrections implement additional curvature required to embed discrete space in quantum space-time. In addition, there are corrections from standard quantum fluctuations of the metric, which are more difficult to compute in loop quantum gravity and have not yet been formulated in a consistent form of hypersurface deformations. (Their main effect is to introduce higher time derivatives [30, 31]. These corrections are therefore close relatives of higher-curvature terms, which do not modify the hypersurface-deformation algebra [32].)

⁵Holonomy corrections are often claimed to be uniquely determined by classical parameters rather than states, especially in cosmology where they are supposed to depend only on the classical density divided by the Planck density. However, the Planck density in this case is chosen ad-hoc, and in general must be replaced by the density of discrete patches, a parameter of the quantum-gravity state. Also holonomy corrections therefore depend on the quantum-gravity state and must be parameterized. Regarding the number of parameters, there is no difference between holonomy corrections and inverse-triad corrections.

When β depends on the metric or extrinsic curvature K_{ab} , our previous arguments about the Poincaré limit as the linear restriction of the hypersurface-deformation algebra still apply. Linear N^a and N lead to a unique subalgebra if h^{ab} and K_{ab} (and therefore β) are spatially constant. In strong quantum regimes, curvature is large and the fields could not be assumed constant. Under these conditions one does not expect a Poincaré algebra to capture space-time properties. A Poincaré description should be valid when quantum effects are not strong and the energy observables are sufficiently small so that back-reaction on space-time can be ignored. Under these conditions, it is safe to assume that the gravitational fields are constant in regions of interest. A distinguished (deformed) Poincaré algebra then follows from the hypersurface-deformation algebra.

However, the deformation is not of the form of non-linear Lie brackets (as part of Hopf algebras) because the modified structure function depends on the phase-space variables in a modified way but does not introduce non-linearities in the generators D and H . Going from Poincaré's Lie algebra to Dirac's Lie algebroid with structure functions has led to a new option for deformations, one not considered before. The expectations of deformed special relativity are therefore not realized, at least not in general. Nevertheless, there are deformations of underlying symmetries which one can try to test as proposed and extensively analyzed in the context of deformed special relativity. Moreover, as we show in what follows, there are regimes in which one can relate phase-space dependent deformations to non-linear algebraic structures.

2.3 Holonomy corrections and energy-dependent deformations

For holonomy corrections, one can, in certain regimes, relate background-dependent deformations in (7) to non-linear algebraic relations. In this case, β depends on extrinsic-curvature components. It has not been possible yet to formulate full non-local holonomies consistently, integrating the connection along curves in arbitrary directions. However, in spherically symmetric models one can implement holonomies along curves on spherical orbits, which then depend on an extrinsic-curvature component K_φ in an angular direction. This component is related to the orbit area $A(x)$ at radial coordinate x by $K_\varphi = -N^{-1}d\sqrt{A}/dt$. The spatial derivative of \sqrt{A} , $k = d\sqrt{A}/dx$, is proportional to the trace of the extrinsic curvature of the orbit 2-sphere in 3-dimensional space.

Before we relate these quantities to observables, we recall features of spherically symmetric models in connection variables [33, 34]. With components of a densitized triad

$$E_i^a \tau^i \frac{\partial}{\partial x^a} = E^x(x) \tau_3 \sin \vartheta \frac{\partial}{\partial x} + E^\varphi(x) \tau_1 \sin \vartheta \frac{\partial}{\partial \vartheta} + E^\varphi(x) \tau_2 \frac{\partial}{\partial \varphi}, \quad (8)$$

whose internal space is written as the Lie algebra $\mathfrak{su}(2)$ with generators $\tau_j = -\frac{1}{2}i\sigma_j$ in terms of Pauli matrices, the spatial metric is

$$ds^2 = \frac{(E^\varphi(x))^2}{|E^x(x)|} dx^2 + |E^x(x)| d\Omega^2. \quad (9)$$

(As a consequence, $A(x) = |E^x(x)|$.) The components E^x and E^φ are canonically conjugate to extrinsic-curvature components K_x and K_φ in

$$K_a^i \tau_i dx^a = K_x(x) \tau_3 dx + K_\varphi(x) \tau_1 d\vartheta + K_\varphi(x) \tau_2 \sin \vartheta d\varphi \quad (10)$$

but not to connection components as in the full theory. (Note however, that K_x is simply a gauge-invariant version of the corresponding connection component A_x with respect to a remnant U(1)-gauge freedom of internal rotations fixing τ_3 .) Using the general relations between extrinsic curvature and time derivatives of the spatial metric, one computes

$$K_\varphi = -\frac{1}{2N\sqrt{|E^x|}} \frac{dE^x}{dt} \quad \text{and} \quad K_x = -\frac{1}{N\sqrt{|E^x|}} \left(\frac{dE^\varphi}{dt} - \frac{E^\varphi}{2E^x} \frac{dE^x}{dt} \right). \quad (11)$$

These relations follow from equations of motion generated by the Hamiltonian constraint and are modified if quantum-geometry corrections of loop quantum gravity are included. The following considerations are independent of these relations.

Holonomy corrections which replace K_φ by $\sin(\delta K_\varphi)/\delta$ (or some other function with related properties) in the Hamiltonian constraint, with a parameter δ that could depend on the triad components, especially E^x , can be implemented consistently [21]. They imply a deformed hypersurface-deformation algebra (7) with $\beta(K_\phi) = \cos(2\delta K_\varphi)$.⁶ More generally, if K_φ is replaced by a function $F(K_\varphi)$, $\beta(K_\phi) = \frac{1}{2} d^2 F^2 / dK_\varphi^2$ [21]. The deformation depends on phase-space variables rather than algebra generators. However, for a specific choice of observers the phase-space variables involved can be related to observables that play the role of algebra generators in the linear limit.

Extrinsic-curvature components determine observables of Brown–York [11] or ADM [12] type: (angular) momentum

$$P = 2 \int_{\partial\Sigma} d^2 z v_b (r_a p^{ab} - \bar{r}_a \bar{p}^{ab}) \quad (12)$$

in direction v^a , measured by an observer who watches the spatial region Σ . (A contribution from a reference metric with barred quantities is subtracted to ensure that energy and momentum vanish in Minkowski space-time.) The integrand depends on the co-normal r_a of the boundary of Σ and the gravitational momentum

$$p^{ab}(x) = \frac{\sqrt{\det h}}{16\pi G} (K^{ab} - K_c^c h^{ab}) \quad (13)$$

canonically conjugate to the spatial metric h_{ab} . In spherical symmetry, we choose a spherical surface $\partial\Sigma$ of constant x , and compute the gravitational momentum using the tensor

$$K_{ab} dx^a \otimes dx^b = K_x \frac{E^\varphi}{\sqrt{|E^x|}} dx \otimes dx + K_\varphi \sqrt{|E^x|} (d\vartheta \otimes d\vartheta + \sin^2 \vartheta d\varphi \otimes d\varphi) \quad (14)$$

⁶The phase-space function $K_\varphi(x)$ is evaluated at the fixed boundary used to define observables, so that no position dependence results.

and its trace

$$K_c^c = K_x \frac{\sqrt{|E^x|}}{E^\varphi} + 2 \frac{K_\varphi}{\sqrt{|E^x|}} \quad (15)$$

(using $K_{ab} = K_a^i e_b^i$ with $E_i^a = e_i^a |\det e_b^j|$). We find that the radial component of the gravitational momentum is proportional to K_φ and independent of K_x :

$$p^{xx} = -\frac{1}{8\pi G} K_\varphi \frac{|E^x|}{E^\varphi} \sin \vartheta. \quad (16)$$

The radial component p^{xx} of the gravitational momentum appears in the linear Brown–York momentum (12) with $v^a = (\partial/\partial x)^a$. We compute

$$P_x = \frac{8\pi g_{xx} p^{xx}}{E^\varphi \sqrt{|E^x|}} = -\frac{1}{G} \frac{K_\varphi}{\sqrt{|E^x|}}.$$

If $\delta \propto |E^x|^{-1/2}$ (corresponding, in the language of holonomy corrections, to lattice refinement [35, 36] with sites of constant size, or a site number per spherical orbit proportional to the orbit area), the combination of variables is exactly what appears in the deformation function $\beta = \cos(2\delta K_\varphi)$. The linear limit of the hypersurface-deformation algebra therefore suggests that the commutator between a boost B_x and a time translation P_0 is deformed by a function depending on the spatial momentum P_x , of the form

$$\{B_x, P_0\} = \cos(\lambda P_x) P_x \quad (17)$$

with a constant λ .

The Brown–York energy is

$$E = -\frac{1}{8\pi G} \int_{\partial\Sigma} d^2z N(\sqrt{\det \sigma} k - \sqrt{\det \bar{\sigma}} \bar{k}) \quad (18)$$

with the induced spatial metric σ_{ab} on $\partial\Sigma$ and the trace k of extrinsic curvature of $\partial\Sigma$ in space, related to $\partial E^x/\partial x$ for a spherical $\partial\Sigma$. This function does not appear in holonomy corrections, and inverse-triad corrections depend on E^x integrated over elementary plaquettes of a discrete state (so-called fluxes) rather than its spatial derivative. If a derivative expansion of the non-local fluxes is used, derivatives of E^x and therefore k will appear, relating the corresponding algebra deformation to the Brown–York energy. However, the relation is not as direct as in the case of holonomy corrections and the radial momentum.

3 Comparison with deformed special relativity

Different realizations of deformed Poincaré algebras are possible, depending on the choice of generators. Moreover, one can remove deformations by non-linear transformations of the generators, giving rise to the question of which expression of the generators should be physically preferred. In fact, it is the choice of generators that determines whether

the algebra is deformed. In our case, using the hypersurface-deformation algebra, the generators have unambiguous physical meaning via the boundary terms used above. Also the bulk terms, taken for matter contributions to the gauge generators which obey the same algebra as the full generators in the absence of derivative couplings, have a clear meaning as the matter energy density

$$\rho_E = \frac{H_{\text{matter}}[N]}{N\sqrt{\det h}} \quad (19)$$

from the matter contribution to $H[N]$, pressure

$$p_E = -\frac{1}{N} \frac{\delta H_{\text{matter}}[N]}{\delta \sqrt{\det h}} \quad (20)$$

and stress

$$S_E^{ab} = -\frac{2}{N\sqrt{\det h}} \frac{\delta H_{\text{matter}}[N]}{\delta h_{ab}} \quad (21)$$

from derivatives of $H_{\text{matter}}[N]$ by the metric, and energy and momentum fluxes

$$J_a^E = \frac{1}{\sqrt{\det h}} \frac{\delta D_{\text{matter}}[N^a]}{\delta N^a} \quad (22)$$

in $D_{\text{matter}}[N^a]$. (See again [9].) As indicated by the labels “E”, these are stress-energy components as measured by Euclidean observers whose worldlines are normal to equal-time hypersurfaces, or co-moving observers in cosmological terminology. Deformed hypersurface-deformation algebras therefore help in identifying physical deformations of fundamental symmetries.

In the linear limit, (7) gives rise to a deformed algebra in which commutators between boosts and time translations, which both follow from $H[N]$, are modified. Commutators in which spatial translations or rotations appear, referring to $D[N^a]$, are undeformed. This behavior is in contrast to the usual representation of the κ -Poincaré algebra [37, 38] in which only the commutator of boosts with spatial translations is deformed,⁷

$$\{B_j, P_k\} = \delta_{jk} \left(\frac{1 - \exp(-2\kappa P_0)}{2\kappa} + \frac{1}{2} \kappa \delta^{mn} P_m P_n \right) - \kappa P_j P_k. \quad (23)$$

No such deformation can result from a gravitational theory which, like loop quantum gravity, implements spatial diffeomorphisms unmodified. Moreover, the κ -Poincaré algebra does not change the relations

$$\{B_j, P_0\} = P_j \quad , \quad \{B_j, B_k\} = -\epsilon_{jkl} R_l \quad (24)$$

which would be deformed in the linear limit of (7); see (17). The relations

$$\{P_\mu, P_\nu\} = 0 \quad , \quad \{R_j, P_0\} = 0 \quad (25)$$

⁷Recall that $\{\cdot, \cdot\}$ is a Poisson bracket, not an anticommutator.

and

$$\{R_j, P_k\} = \epsilon_{jkl} P_l \quad , \quad \{R_j, R_k\} = \epsilon_{jkl} R_l \quad , \quad \{R_j, B_k\} = \epsilon_{jkl} B_l \quad (26)$$

are undeformed in both cases.

The more-general parameterizations of generalized Poincaré algebras in [39] allows a version in agreement with the deformation found here. The ansatz made there leaves rotation generators undeformed, just as we need it to make contact with our deformations. Boosts are deformed by a parameterization of a boost operator deviating from $\hat{x}_i \hat{p}_0 - \hat{x}_0 \hat{p}_i$. It follows from Eq. (16) in [39] that an undeformed commutator of a boost with a spatial rotation, as implied by the unmodified (5), requires that only the second part of the boost operator is modified, to $\hat{x}_i \hat{p}_0 - \beta \hat{x}_0 \hat{p}_i$. No extra terms quadratic in momenta and linear in position operators, as possible more generally, are allowed. Qualitatively, the commutators (16) and (18) in [39] then agree with the deformations obtained here in the linear limit of (7). However, while the deformed algebras of observables agree, the required form of boost generators is not compatible with the representation of κ -Minkowski space discussed in [39].⁸ If a compatible representation can be found, it would serve as a candidate for a quantum space-time model corresponding to a deformed algebra (7).

4 Discussion

Using the hypersurface-deformation algebra underlying canonical gravity, we have extended deformed special relativity to deformed general relativity, with several advantages:

- The deformation is derived from loop quantum gravity, which in recent years has produced several mutually consistent versions of deformed off-shell constraint algebras. The derivation clearly shows how quantum geometry gives rise to deformed fundamental symmetries.
- The generators of deformed hypersurface-deformation algebras have unambiguous physical meaning as boundary observables or stress-energy components. The deformation does not depend on one's choice of generators.
- The deformation originally appears in position space, on which the classical structure functions of the algebra are defined. A corresponding momentum-space version exists only in certain regimes and depends more sensitively on which observer one refers to. Deformed special relativity, by contrast, is formulated in momentum space, and it is not only difficult to transform to position space but also problematic because of locality problems [40, 41, 42].
- Deformations of the hypersurface-deformation algebra in off-shell loop quantum gravity depend on the discreteness scale of quantum space or on the local extrinsic curvature, not on the mass of macroscopic bodies relative to the Planck mass. The

⁸We are grateful to Anna Pachol for pointing this out to us.

“soccer-ball problem” does not occur, in a way that resembles proposed solutions in deformed special relativity [4].

Given these promising indications, the task of completing the understanding of off-shell algebras in loop quantum gravity receives increased prominence. All derivations so far have shown that the form (7) of deformed algebras and their deformation functions β appears to be universal, but they remain incomplete. Especially the inclusion of full holonomies with integrations of the gravitational connection along curves remains a challenge. The integrations involved may suggest non-local deformations of the Poincaré algebra in the linear limit.

The relation between deformed special relativity and deformed general relativity is not very direct. Nevertheless, the more-general viewpoint developed here supports the basic ideas and motivations behind deformed special relativity, even if there are differences in concrete realizations. Some of the problems discussed extensively in deformed special relativity do not appear in deformed general relativity or can easily be solved, but a full analysis may still reveal new issues, including observational ones. One may hope that the detailed methods developed and used to scrutinize deformed special relativity can be applied to deformed general relativity to put stringent tests on the underlying theory of loop quantum gravity. Even though the derivation from loop quantum gravity makes use of several assumptions and approximations, the broad consensus and universality reached by all existing computations of off-shell constraint algebras, be it by effective or operator methods, shows that loop quantum gravity can be ruled out if its version of deformed general relativity is ruled out.

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